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Variability and upper bounds for maximum ground level concentrations

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The properties of the various factors which arise in the conventional formula for maximum hourly mean ground level concentration (max. g.l.c.) after fixing mean wind speed and source strength are studied with a view to assessing variability and comparing with results from the Tilbury field trial. The analysis indicates some ambiguity in defining vertical spread for dispersion from tall stacks and suggests that the formula might be more profitably rearranged. A natural rearrangement is pointed out and this leads to a simple upper bound for max. g.l.c. in an unbounded atmosphere which is only 20% larger than the familiar result for a uniform atmosphere. This result is obtained using the diffusion equation with simple power laws for wind speed and eddy diffusivity. The general conservation equation is then considered and this leads to a specific definition of mean wind speed below source level and the indication of a general upper bound some 50% larger than the uniform atmosphere value provided certain reasonable conditions are met. The practical implications of these results are discussed and the extra effects introduced by stable layers are pointed out.

1. Introduction

One of the most important results to emerge from the experimental field trial at Tilbury (Moore, this volume, p. 107) is that the simple Gaussian model of a plume gives very reasonable results for hourly mean values of maximum ground level concentration (max. g.l.c.). In the majority of cases it tends to slightly overestimate the max. g.l.c. and in general appears to be in error by a factor in the range 0.2 to 2.0. To understand these results more fully a series of calculations have been made in parallel with the experimental work aimed at answering the following questions. Why (in the deductive, not inductive, sense) do simple assumptions regarding vertical concentration distributions give reasonable results for max. g.l.c.? Has there been any cancelling of errors, for example? (The answers to these questions have application in numerous other fields as well as air pollution, e.g. heat transfer, building wakes, cooling water studies, etc.) What variability about the Gaussian result does theory indicate and how does this compare with observation? Is there an upper limit to the value that the max. g.l.c. can achieve for the case of dispersion into an unbounded atmosphere with reflexion at the ground? If so, is this limit of practical value and how is it affected by inversion layers? Answers to the last two questions emerge most naturally by considering the first three, but partly for reasons of brevity and partly because they are probably the more important questions with the more immediate practical implications, the present paper will concentrate on the last two. Detailed results on the others will be published subsequently. The scheme of the paper is therefore as follows. Section 2 examines the factors that occur in the conventional formula for max. g.l.c. after wind speed and source strength have been fixed and points out that they do have appreciable variabilities but in opposing directions, indicating that a rearrangement of factors is more illuminating. Section 3 considers the rearrangement for the case when wind speed and eddy diffusivity vary as simple power laws in height above ground and shows that an upper bound exists for this case which is only 20 % above the classical result for the uniform unbounded atmosphere. It also emphasizes the ambiguity introduced by retaining a term in root mean square vertical spread in cases where variation of atmospheric properties can occur across the total depth of plume at the point of max. g.l.c. Section 4 gives an analysis which indicates that an upper bound for the unbounded atmosphere exists in general if it is possible to define an equivalent eddy diffusivity in the manner described and this would appear possible in most cases of interest. Section 5 gives a brief description of some results on the variability caused by stable layers above or below the source level and the effect that these have upon the upper bound.

2. Similarity approximations for maximum ground level concentration

The majority of working formulae for max. g.l.c. (for example, Pasquill 1962) are based upon the following three assumptions:

(i) That the steady state concentration distribution, $C(x,y,z)$,† can be written as

$$C(x,y,z) = \frac{X(x,z)}{\sigma_y(x,z)\sqrt{(2\pi)}} \exp\left[-\frac{y^2}{2\sigma_y^2(x,z)}\right], \tag{1}$$

where $X(x,z)$ is the integral crosswind concentration (or line source function) and $\sigma_y(x,z)$ is the effective crosswind standard deviation. Hereafter the ground level value, $\sigma_y(x,0)$, will be abbreviated to $\sigma_y(x)$.

(ii) That the correct equation representing conservation of the rate at which effluent crosses any vertical plane normal to the mean wind direction, i.e.

$$\int_0^\infty u(z)\, X(x,z)\, \mathrm{d}z = Q, \tag{2}$$

can be replaced by a relation like

$$\int_0^\infty X(x,z)\, \mathrm{d}z = Q/\bar{u}. \tag{3}$$

Here $u(z)$ is the wind speed at height z, $\bar{u}$ is some unspecified mean wind speed (a subject which will be mentioned again later), and Q is the total rate of emission of effluent by the source.

(iii) That the 'shape' of the concentration distribution in the vertical remains fixed at all distances downwind and that vertical distances from the source level, $z = H$, can be scaled according to downwind distance only (the 'similarity' assumptions). Allowing perfect reflexion at the ground ($z = 0$) this means that $X(x,z)$ is assumed to have the form

$$X(x,z) = \frac{A}{\sigma_z(x)}\left\{f\left[\frac{z-H}{\sigma_z(x)}\right] + f\left[-\frac{(z+H)}{\sigma_z(x)}\right]\right\}, \tag{4}$$

where A is a normalization constant and $\sigma_z(x)$ is a vertical spreading coefficient often taken to be equal to the r.m.s. vertical spread of serially released particles. In general $f(Z)$ is arbitrary and not symmetric about $Z = 0$. For the classical Gaussian model $f(Z) = \mathrm{e}^{-\frac{1}{2}z^2}$.

With these assumptions the evaluation of max. g.l.c., $C_{\max}$, as the maximum value of $C(x,0,0)$ is straightforward and leads to a result which can be written in the form:

$$\left.\begin{aligned} C_{\max} &= \frac{NQ}{\bar{u}H^2}\left[\frac{\sigma_z(x)}{\sigma_y(x)}\right]_{x_{\max}}, \\ \sigma_z(x_{\max}) &= \gamma H. \end{aligned}\right\} \tag{5}$$

Here $x_{\max}$ is the downwind distance at which the max. g.l.c. occurs, N is a pure number which depends only upon the 'shape' function $f(Z)$, i.e.

$$N = (2/\pi)^{\frac{1}{2}} \max\{Z^2 f(-Z)\} \Big/ \int_{-\infty}^{\infty} f(Z)\, \mathrm{d}Z, \tag{6}$$

† The conventional rectangular coordinate system (x, y, z) is used throughout, with x in the downwind direction, y in the crosswind direction, and z in the vertical. The origin is vertically below the source point which has coordinates $(0, 0, H)$. H represents stack height plus plume rise.

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and γ is similar number which also depends upon the assumed behaviour of $\sigma_y(x)$ and $\sigma_z(x)$. Equations (5) are exact for arbitrary power law variations of $\sigma_y(x)$ and $\sigma_z(x)$ and so include the analyses of Wippermann & Klug (1962) and Moore (1967) as special cases. In the general case they give a close approximation. For the Gaussian model $N = 2/e\pi$. It is frequent practice to assume that σ_z/σ_y is independent of distance (a point which will be commented upon later) in which case $\gamma = 1/\sqrt{2}$. The 'classical' model may be described as that which uses these simple values for N and γ together with the assumption that σ_z/σ_y is equal to $1/2$ for hourly averages.

It is proposed to examine equations (5) when source strength, Q, and wind speed, $\bar{u}$, are held fixed, so that any variability in $C_{\max}$ arises principally from changes in vertical concentration profile, etc. As plume rise for moderately large plants is essentially determined by $\bar{u}$ this means that the factors H and $Q/\bar{u}H^2$ are taken to be constant throughout. The remaining factors to be considered are N and $(\sigma_z/\sigma_y)_{x_{\max}}$ or some preferable subdivision of their product.

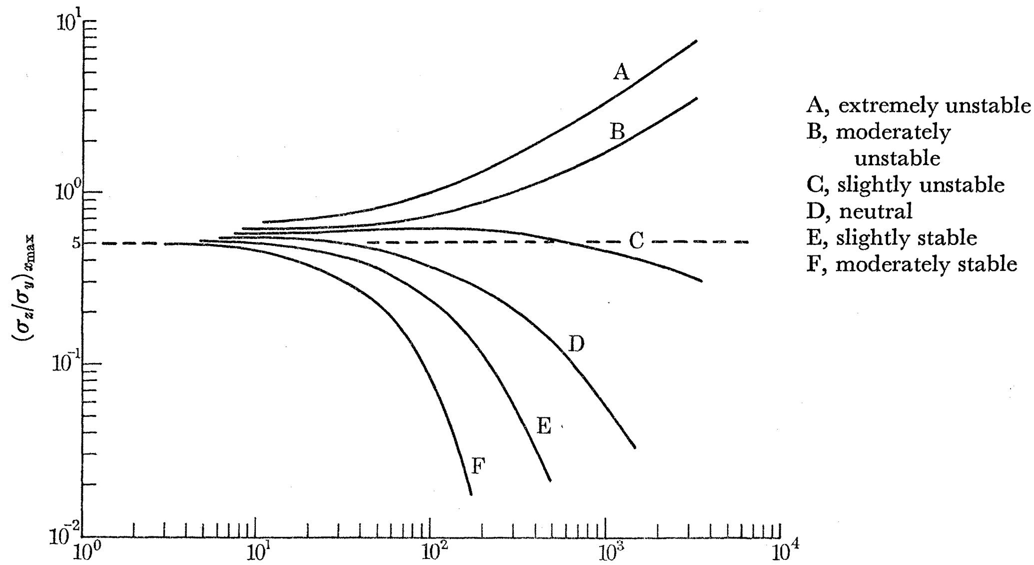

FIGURE 1. Variation of $(\sigma_z/\sigma_y)_{x_{\max}}$ with source height and stability.

Pasquill (1962 pp. 197–8) indicates that variations of up to 30 to 50 % might be expected in the value of N according to the form assumed for $f(Z)$. The forms that he adopts are quite empirical and are based upon variants of exponential or Gaussian functions. This approach in itself could be a limiting factor. A more instructive, but still somewhat empirical approach, is to use the results of diffusion theory given by O. F. T. Roberts (unpublished; see Sutton 1953) and Smith (1957) and this will be given in the next section. This has the advantage of bringing out the difficulty in defining, unambiguously, a value for $\sigma_z(x)$ in an atmosphere where wind speed and turbulence characteristics vary with height.

Leaving this difficulty on one side for the moment, the variability to be expected in practice from the factor $(\sigma_z/\sigma_y)_{x_{\max}}$ can be readily calculated from the curves for $\sigma_y(x)$, $\sigma_z(x)$ suggested by Pasquill (1962, p. 209; see also Gifford 1968) for his stability categories A–F (defined above). These data allow $\sigma_z(x)/\sigma_y(x)$ to be plotted as a function of $\sqrt{2}\,\sigma_z(x)$ for each category and at the point of max. g.l.c. the ordinate is equal to the required quantity whilst the abscissa is approximately equal to the source height H. The results are given in figure 1. For small source heights (say, $H < 20$ m) the values cluster about the 'classical' value of 0.5, the principal departure being towards higher values in unstable conditions. For the Tilbury and Northfleet plants the source height lies mainly in the range 200 to 300 m corresponding to values of $(\sigma_z/\sigma_y)_{x\max}$ from 10^{-2} in stable conditions to values greater than unity in very unstable or convective conditions. It would be temptingly simple therefore to explain the observed variability in $C_{\max}$ at Tilbury by keeping N fixed at the Gaussian value $2/e\pi$ and allowing $(\sigma_z/\sigma_y)_{x\max}$ to range over the above values. However, there are a number of objections to this solution.

(*a*) On a number of occasions when the Tilbury plume appears to satisfy the similarity requirements, the plume from the nearby Northfleet plant with the taller stacks (150 m as opposed to 100 m) does not, i.e. the measured $C_{\max}$ does not tally with the calculated value.

(*b*) At the point of max. g.l.c. at Tilbury, the depth of the plume is some 400 m. It would be tempting providence to ignore changes in wind speed, turbulence characteristics, etc., over such depths and it is therefore unlikely that the Gaussian value for N is correct.

(*c*) Immediately one introduces changes of atmospheric properties with height the amount of vertical dispersion, i.e. σ_z, becomes a function of the height of release. It also depends upon the form assumed for $f(Z)$ as does N. Hence for tall stacks, corresponding to $H > 30$ m, the variability associated with N and $(\sigma_z/\sigma_y)_{x\max}$ cannot be considered as separate issues.

These points all refer to the similarity assumption (iii) and in particular to the definition of $\sigma_z(x)$. This prompts the suggestion that equations (5) might more usefully be combined to read

$$C_{\max} = \frac{N'Q}{\bar{u}H\sigma_y(x_{\max})}, \tag{8}$$

where N' is again a pure number and the relation by which one obtains $x_{\max}$ is for the moment left open. Two immediate advantages of equation (8) are that the Pasquill data on $\sigma_y(x)$ are probably valid for tall stacks and show less variability than the $\sigma_z(x)$ data, and secondly that $\sigma_y(x_{\max})$ is directly deducible from ground level surveys of the Tilbury type. Values of $\sigma_y(x)$ estimated from the Tilbury data suggest an almost linear dependence on x in agreement with the Pasquill data. It should also be pointed out that the variation of $C_{\max}$ with sampling time comes principally from the variation of $\sigma_y(x_{\max})$ with sampling time (provided this time is greater than about 5 min) so that equations (5) and (8) are equivalent in this respect.

The values of N and N' will now be evaluated on the basis of non-similarity assumptions.

3. The values of N and N' for simple shear flow

It is well known that the only practical method available for estimating the effects on concentration distributions of varying mean wind speed and turbulence properties with height is the diffusion equation approach. There are objections to using this approximation in the present context (see Gifford 1968), but any reasonable improvement entails a disproportionate increase in complexity (see for example, Roberts 1961), probably for little return in accuracy. It will therefore be adopted in what follows. The starting point is the steady state continuity equation for the integrated crosswind concentration, $X(x, z)$, i.e.

$$u(z)\frac{\partial X}{\partial x} = \frac{\partial}{\partial z}\left(K_z(z)\frac{\partial X}{\partial z}\right), \tag{9}$$

where $K_z(z)$ is the usual vertical eddy diffusivity. Equation (9) replaces assumptions (ii) and (iii) of § 2. The boundary conditions to be satisfied by $X(x, z)$ are conventional, i.e. that $X(x, z)$ vanishes as either x or z tend to infinity, that it should represent a line source of strength Q along the line $x = 0$, $z = H$, and that the net flux, $(-K_z\partial X/\partial z)$, should vanish at ground level, $z = 0$.

Following O. F. T. Roberts we adopt the simple power law relations

$$u(z) = u(H)\left(\frac{z}{H}\right)^m, \quad K_z(z) = K_z(H)\left(\frac{z}{H}\right)^n,$$

where for the moment the only restrictions upon m and n are that m should be positive. The solution for the ground level concentration, $X(x, 0)$ is then found to be

$$X(x,0) = \frac{2+m-n}{\Gamma(s)}\,\frac{Q}{u(H)\,H}\left(\frac{x}{l_{mn}}\right)^{-s}\exp\left(\frac{-l_{mn}}{x}\right), \tag{10}$$

where

$$\left.\begin{aligned} s &= (1+m)/(2+m-n),\\ l_{mn} &= u(H)\,H^2/(2+m-n)^2\,K_z(H(.\end{aligned}\right\} \tag{10a}$$

To obtain the max. g.l.c. some assumption has to be made about the behaviour of $\sigma_y(x)$. The Pasquill data mentioned in § 2 and the ground level SO_2 patterns observed at Tilbury are both consistent with the relation

$$\sigma_y(x) \propto x^q, \tag{11}$$

where q is slightly less than or almost equal to unity. Then from equations (1), (10) and (11) it follows that

$$C_{\max} = \frac{(2+m-n)\,(s+q)^s\,e^{-(s+q)}}{\sqrt{(2\pi)}\,\Gamma(s)}\,\frac{Q}{u(H)\,H\sigma_y(x_{\max})}, \tag{12}$$

and

$$x_{\max} = l_{mn}/(s+q). \tag{13}$$

The equation for $C_{\max}$ thus arises quite naturally in the form of (8) rather than (5) as will will be further demonstrated in the next section. To introduce an effective value of $\sigma_z(x)$ so as to put (12) into the form (5) is only straightforward and unambiguous in the case $q = \frac{1}{2}$. The assumption that the value of N so obtained is accurate for general values of q in the range 0.5 to 1.0 is reasonable in that if N were sensitive to q it would indicate a strong interaction between crosswind and vertical diffusion over distances of order $x_{\max}$, and this is unlikely on physical grounds. Hence one can adopt the uniform atmosphere diffusion form for $\sigma_y(x)$, i.e.

$$\sigma_y^2(x) = 2K_y(H)\,x/u(H),$$

where $K_y(H)$ is the crosswind eddy diffusivity at the source level. Then (12) can be written as

$$C_{\max} = \frac{N_{m,n}\,Q}{u(H)\,H^2}\left[\frac{K_z(H)}{K_y(H)}\right]^{\frac{1}{2}},$$

where

$$N_{m,n} = \frac{2}{e\pi}\left[1+\tfrac{1}{2}(m-n)\right]^2\frac{\sqrt{\pi}}{\Gamma(s)}\,(s+\tfrac{1}{2})^{s+\frac{1}{2}}\exp\left[-\frac{m+n}{2(2+m-n)}\right]. \tag{14}$$

Values of the factor F equal to N_{mn} divided by the classical Gaussian value $2/e\pi$ are given in figure 2 in isopleth form. They increase monotonically with m for a fixed value of n and decrease monotonically as n increases for a fixed value of m. In the majority of practical cases m will not exceed 0.2 so that F lies in the range 0.5 to 1.5 for the range of values shown in figure 2. The curve for constant horizontal shear stress (i.e. $n = 1-m$, shown dashed) is a case where m and n change simultaneously to give the most rapid increase possible in F. However, for $m < 0.2$ this condition only gives F values in the range 0.5 to 0.8, i.e. smaller than the uniform atmosphere case $m = 0 = n$. The largest values of F occur for negative values of n, i.e. in cases where stability increases with height (provided that stability is in some sense synonymous with diffusivity). The limitations of the model make it difficult to be too specific but it certainly appears from figure 2 that cases where N is large correspond in figure 1 to areas where $(\sigma_z/\sigma_y)_{x_{\max}}$ is small, and vice versa. This behaviour suggests the possibility that the product has a maximum.

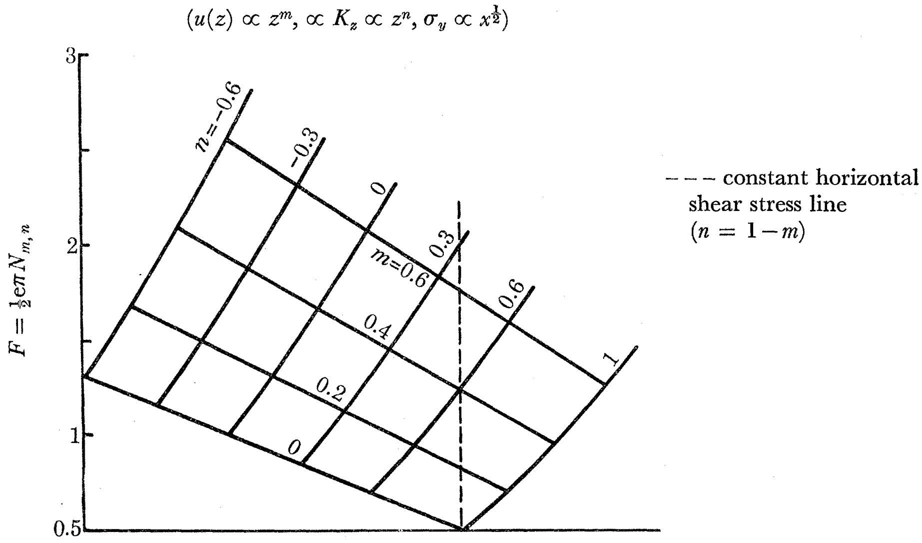

FIGURE 2. Isopleths of the variability factor F for the simple shear flow model.

The form of (14) and the analysis of the next section indicate that the first step towards obtaining a more satisfactory formulation is to define the mean wind speed below the source level by the simple integral relation

$$\bar{u} = \frac{1}{H}\int_0^H u(z)\,dz. \tag{15}$$

In this case $u(H)$ becomes equal to $(1+m)\,\bar{u}$ for the power law variation. Then, taking $q = 1$ as the most reasonable value for q, (12) takes the form of (8) with N' given by

$$N' = \frac{(1+s)^s\, e^{-(1+s)}}{\sqrt{(2\pi)}\,\Gamma(1+s)}, \tag{16}$$

which is purely a function of the parameter s. Values computed from (16) are given in figure 3 where it is at once evident that N' has a maximum value of $1/e\sqrt{(2\pi)} = 0.147$ for $s = 0$ (corresponding to $n \to -\infty$) and decreases monotonically as s increases. The boundary layer with

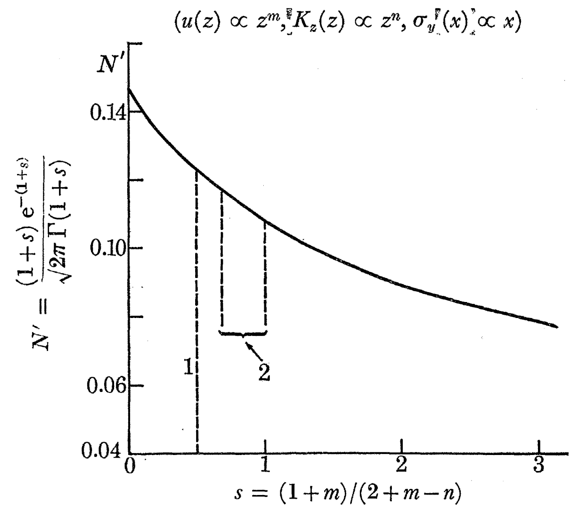

FIGURE 3. Values of N' for the simple shear flow model. Dashed line 1 gives s for a uniform atmosphere where $m = 0 = n$, while 2 gives the range of s for constant horizontal shear stress ($K_z\, du/dz$).

constant horizontal stress considered by Smith (1957) corresponds to s in the range $\frac{2}{3}-1$ which in turn corresponds to values of N' within ± 5 % of the value 0.11. The uniform atmosphere case, $m = 0 = n$, corresponds to $s = 0.5$ and $N' = 0.123$. The maximum value at $s = 0$ is only 20 % larger than this, a perhaps surprising and important result which reflects upon the question as to why simple formulae give good results. The Tilbury data seem to support this conclusion. It is not possible to be more definite owing to experimental uncertainties (themselves of order 20 %) and the difficulty in deciding from the meteorological data as to whether a stable layer is influencing a particular result or not.

4. The possibility of a general upper limit to C_{max} for the unbounded atmosphere

Historically, the present work began with an attempt to devise a perturbation or variational method for solving (9) when $u(z)$ and $K_z(z)$ are fairly general functions of z. This would throw light on such questions as the sensitivity of C_{max} to variations in profiles of $u(z)$ and $K_z(z)$ and the results of § 3 could be used as an exact check. A perturbation method has been developed and will be mentioned briefly later. The detailed results will be published elsewhere. The first approximation in this method is taken as an 'equivalent uniform atmosphere' function, that is to say a solution to the problem where $K_z(z)$ (or rather the function corresponding to $K_z(z)$ after a change in height scale) is replaced by a suitably chosen constant. This approach leads to the following results:

(i) If it is possible to choose this constant so that the max. g.l.c. is unchanged then C_{max} for the elevated point source in an unbounded atmosphere has an upper bound.

(ii) The second order perturbation indicates that this choice is usually possible and that the first approximation is fairly accurate. The last point of course explains why the simple formulae give good estimates of C_{max}.

The results of the last section confirm the existence of the upper bound when $u(z)$ and $K_z(z)$ are simple monotonic functions of z. In general one would expect the line source function at ground level, $X(x, 0)$, to have a maximum value since it vanishes when $x = 0$ and tends to zero again as x tends to infinity. The only case when $X(x, 0)$ does not vanish at infinity is when a perfectly reflecting boundary is present above the source. Hence as $\sigma_y(x)$ is a monotonically increasing function of x for all x the function $X(x, 0)/\sigma_y(x)$ must also have a maximum which always lies upwind of the maximum of $X(x, 0)$. Therefore:

$$C_{max} \equiv \max\{C(x, 0, 0)\}$$
$$= \frac{1}{\sqrt{(2\pi)}} \max\left\{\frac{X(x, 0)}{\sigma_y(x)}\right\}$$
$$\leqslant \frac{\max\{X(x, 0)\}}{\sqrt{(2\pi)}\,\sigma_y(x_{max})}, \qquad (17)$$

where x_{max} is again the true distance where $C(x, 0, 0)$ has its maximum.

To be specific about the numerator on the right hand side of (17) requires deeper examination of (9), or better still of the general continuity equation

$$u(z)\frac{\partial X}{\partial x} = -\frac{\partial F}{\partial z}, \qquad (18)$$

where $F(x, z)$ is the net vertical flux of material. If item (i) above is valid then it may well be valid generally and not just in the diffusion approximation. As the form of the analysis is the same whichever equation is used, (18) will be considered hereafter.

The first step is to scale the vertical coordinate z so as effectively to replace the mean wind speed profile, $u(z)$, by a constant value. It is also convenient on the new scale to keep ground level as zero but to non-dimensionalize so that the source occurs at unit height. We therefore define $\zeta(z)$ by

$$\zeta(z) = \int_0^z u(z')\,\mathrm{d}z' \Big/ \int_0^H u(z)\,\mathrm{d}z, \tag{19}$$

and a mean wind spread, $\bar{u}$, by (15). Then (18) becomes

$$\bar{u}\frac{\partial X}{\partial x} = -\frac{1}{H}\frac{\partial F}{\partial \zeta}, \tag{20}$$

and on the ζ-scale the conservation relation (2) becomes

$$\int_0^\infty X(x, \zeta)\,\mathrm{d}\zeta = Q/\bar{u}H. \tag{21}$$

Equation (21) is of the required form (3) except that now $\bar{u}$ is specifically defined according to (15). It is also evident from (1) (with $y = 0$) and (21) that the relation for $C_{\max}$ will occur naturally in the form of (8).

The next step is to introduce an arbitrary positive constant, $\bar{k}$, by writing (20) in the form

$$H\bar{u}\frac{\partial X}{\partial x} = \bar{k}\frac{\partial^2 X}{\partial \zeta^2} - \frac{\partial}{\partial \zeta}\left[F + \bar{k}\frac{\partial X}{\partial \zeta}\right]. \tag{22}$$

Here $\bar{k}$ has the dimensions of a velocity but in the diffusion approximation the dimensional analysis can be taken further to define the natural unit of length in the x-direction similar to l_{mn} in (10a). If the last term in (22) is ignored, the solution can be written down immediately in terms of the usual Gaussian type expressions with $(\zeta \pm 1)$ replacing $(z \pm H)$. Similarly, if the last term is replaced by the unit point source expression $\delta(x - x_0)\,\delta(\zeta - \zeta_0)$ the solution, i.e. the Green function $G(x, \zeta | x_0, \zeta_0)$, is the same function (apart from normalization) only with $(x - x_0)$ and $(\zeta \pm \zeta_0)$ replacing x and $(\zeta \pm 1)$, respectively. It also vanishes for $x < x_0$. Treating the last term in (22) as a source term the general solution can therefore be written in the form

$$X(x, \zeta) = QG(x, \zeta | 0, 1)/\bar{u}H - \iint \frac{\partial}{\partial \zeta_0}\left[F(x_0, \zeta_0) + \bar{k}\frac{\partial X(x_0, \zeta_0)}{\partial \zeta_0}\right] G(x, \zeta | x_0, \zeta_0)\,\mathrm{d}x_0\,\mathrm{d}\zeta_0.$$

By using the boundary conditions, the last term integrates by parts to give finally

$$X(x, \zeta) = QG(x, \zeta | 0, 1)/\bar{u}H + \int_0^x \mathrm{d}x_0 \int_0^\infty \left[F(x_0, \zeta_0) + \bar{k}\frac{\partial X(x_0, \zeta_0)}{\partial \zeta_0}\right] \frac{\partial G(x, \zeta | x_0, \zeta_0)}{\partial \zeta_0}\,\mathrm{d}\zeta_0. \tag{23}$$

In principle the value of $\bar{k}$ can now be chosen so that the integral term in (23) vanishes at the point of max. g.l.c. provided: (a) that the double integral which multiplies $\bar{k}$ in this term does not vanish for $\zeta = 0$ and $x = x_{\max}$, and (b) that the value of $\bar{k}$ so obtained is positive. The perturbation theory [where $X(x_0, \zeta_0)$ in the integral term is replaced by $QG(x_0, \zeta_0 | 0, 1)/\bar{u}H$] indicates that ($a$) and ($b$) will be satisfied in most cases of interest. Hence on putting $x = x_{\max}$, $\zeta = 0$, and using

$$G(x, \zeta | 0, 1) = (\pi\xi)^{-\frac{1}{2}}\left\{\exp\left[-\frac{(\zeta - 1)^2}{\xi}\right] + \exp\left[-\frac{(\zeta + 1)^2}{\xi}\right]\right\}, \tag{24}$$

where $\xi = 4\bar{k}x/H\bar{u}$, one obtains finally from (1) that

$$C_{max} = \frac{P(\xi_{max})}{\pi\surd e}\frac{Q}{\bar{u}H\sigma_y(x_{max})}. \quad (25)$$

Here ξ_{max} is equal to $4\bar{k}x_{max}/H\bar{u}$ and the function $P(\xi)$ is defined by

$$P(\xi) = \left(\frac{2}{\xi}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}-\frac{1}{\xi}\right).$$

It is in fact the shape function for the ground level concentration from a line source in a uniform unbounded atmosphere normalized to unity at its maximum value, which occurs when $\xi = 2$. To calculate $\bar{k}$ and x_{max} in general is a complex procedure because although the source term in (23) vanishes for $x = x_{max}$ its derivative with respect to x does not. However, irrespective of the particular values of $\bar{k}$ and x_{max} that might arise in practice (25) shows that

$$C_{max} \leqslant \frac{1}{\pi\surd e}\frac{Q}{\bar{u}H\sigma_y(x_{max})}, \quad (26)$$

giving a general upper bound of $1/\pi\surd e = 0.193$ to the value of N'. This is 30 % higher than the value 0.147 obtained from the diffusion model of the last section and 50 % higher than the value of N' in the uniform atmosphere approximation.

5. Effects due to a bounded atmosphere

The Green function used above contains no reference to the fact that vertical diffusion may be almost completely halted at some height due to a layer of stable air. To include this effect the boundary condition of flux tending to zero as height increases indefinitely could be modified to make the flux vanish at a specific height, say z_0. This would imply perfect reflexion at this level which for all practical purposes is sufficient as the later results of this section will show. As indicated earlier (Scriven 1967), this overestimates the effect of stable layers and has maximum

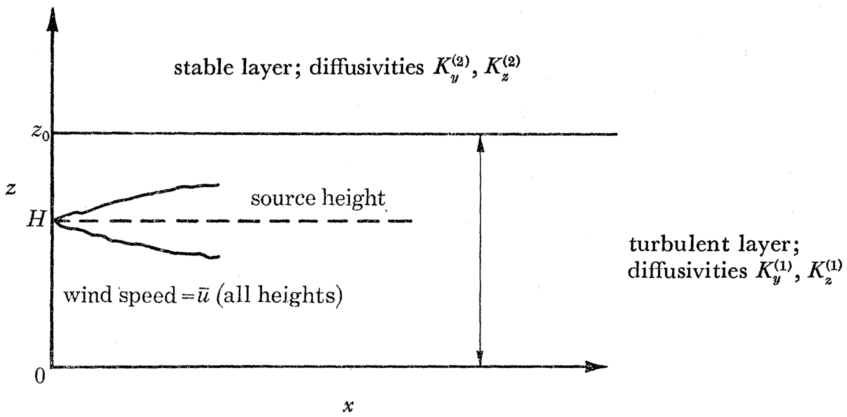

Figure 4. Diagram illustrating the simple two-layer atmospheric model used in § 5.

effect when the layer height z_0 equals the source height H, the max. g.l.c. then being increased by a factor of two. As this factor will apply to the Green function $G(x, \zeta|0, 1)$ the absolute maximum value of N' is therefore equal to $2/\pi\surd e = 0.386$. This is slightly smaller than the value $1/\surd(2\pi) = 0.399$ obtained by assuming that the effluent from the line source disperses with speed $\bar{u}$ and uniform concentration between two perfectly reflecting barriers distant H apart. This must of course overestimate the true state of affairs as the ground level concentration in this situation will always be smaller than concentration up aloft except at large distances from the source.

To assess the effects of stable layers in rather more detail some calculations have been done for a point source on the basis of the simple two-layer diffusion model illustrated in figure 4. Diffusion in the crosswind direction is now included and the transition from a lower turbulent region to an upper stable one is assumed to take place at a single height, z_0, which can be either above or below the source. The wind speed is taken to be constant at all heights, and the cross-wind and vertical eddy diffusivities K_y, K_z, respectively, are taken to have constant but different values in each layer. Details of the analytic results obtained will be given elsewhere. Essentially they consist of solving the diffusion equations

$$\bar{u}\frac{\partial C}{\partial x} = K_y^{(i)}\frac{\partial C^2}{\partial y^2} + K_z^{(i)}\frac{\partial^2 C}{\partial z^2}, \quad (i=1, 2)$$

for the two regions by the moments method (Smith 1957), calculating the position and value of the max. g.l.c., and normalizing this value to the max. g.l.c. for the lower layer alone, i.e. when $z_0 = \infty$. The result is effectively the quantity F defined earlier and it depends solely upon the three dimensionless ratios

$$K_y^{(2)}/K_y^{(1)}, \quad K_z^{(2)}/K_z^{(1)}, \quad z_0/H.$$

Two sets of calculations were performed, one with $(K_y^{(2)}/K_y^{(1)})$ set equal to unity and a second with it set equal to $(K_z^{(2)}/K_z^{(1)})$ (the more likely situation in practice). The difference obtained in the

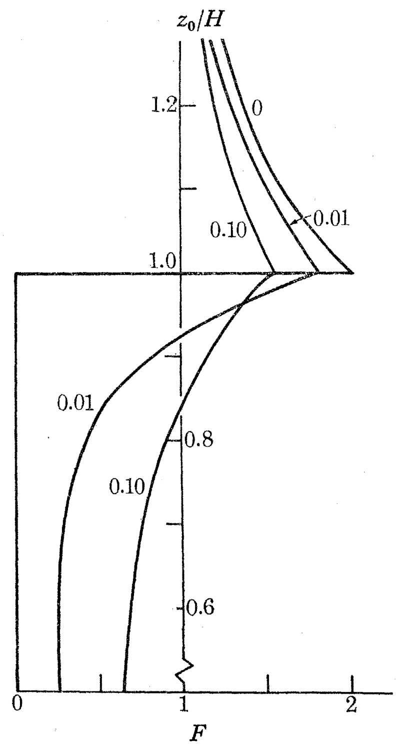

FIGURE 5. Dependence of $F = C_{max}(z_0)/C_{max}(z_0=\infty)$ on z_0/H and $K_z^{(2)}/K_z^{(1)}$ for the case where $K_y^{(2)}/K_y^{(1)} = K_z^{(2)}/K_z^{(1)}$. The numbers on the curves are the values of $K_z^{(2)}/K_z^{(1)}$. This value is 0 for a perfect reflector.

F values was quite small in the majority of cases, the exception being when z_0/H was less than about 0.2. This seems to confirm the earlier supposition that max. g.l.c. is not sensitive to the vertical profile of crosswind eddy diffusivity nor is there any significant interaction effect between crosswind and vertical diffusion. Hence only the results for the second set of calculations are given in figure 5. The earlier calculations already mentioned (Scriven 1967) indicate that in practice the diffusivity ratio across stable layers might be expected to lie somewhere in the range 10^{-2} to 10^{-1}.

The results given in figure 5 suggest the following:

(i) Stable layers at heights greater than about $1\frac{1}{4}$ source heights have no significant effect upon C_{max} as predicted earlier (Scriven 1967).

(ii) C_{max} is very sensitive to the position of the layer when it falls in the range $0.8H$ to $1.2H$. As z_0 decreases through this range the value of F increases from 1.2 to 1.3 to a peak value 1.8 to 2.0 and then decreases rapidly to a value between 0.2 and 0.5 as the layer height decreases further.

(iii) The maximum value of F is equal to

$$2/[1+(K_z^{(2)}/K_z^{(1)})^{\frac{1}{2}}],$$

so that it can never exceed two in theory (corresponding to a perfectly reflecting layer just above the level at which the plume levels out) and probably 1.8 in practice.

(iv) It follows from (ii) that F is sensitive in reality to the mechanics of penetration of an inversion which throws some doubt upon the present oversimplified model. This may affect the details but not the overall results.

(v) If the plume is released into or penetrates the stable region (probably corresponding to $z_0 = 0.2H - 0.8H$) a low value of F is obtained, around 0.2. The downwind position of max. g.l.c. is also considerably increased. Many of these occasions could be missed altogether with a limited number of ground based recorders.

6. Conclusions

The properties of the various factors which arise in the conventional formula for max. g.l.c. after fixing mean wind speed and source strength have been studied with a view to assessing their variability and comparing with the results from the Tilbury experiment. Three general points seem to emerge:

(1) There is some difficulty in assigning an unambiguous definition to the vertical spread (i.e. σ_z) in the case of dispersion from tall stacks ($>$ 20 m).

(2) Factors often assumed constant (e.g. σ_z/σ_y) have quite large variabilities when taken over the spectrum of meteorological conditions.

(3) Different factors vary in opposing directions thus cutting down the overall variability of max. g.l.c.

Points (1) and (3) suggest a rearrangement of the formula and this is carried out when wind speed and vertical eddy diffusivity are assumed to be simple power laws in height above ground. The revised arrangement indicates an upper bound for max. g.l.c. when dispersion takes place in an unbounded atmosphere, a bound which is only 20 % greater than the value given by the familiar uniform atmosphere model. The theory is then extended to the general case, still with an unbounded atmosphere, and this again indicates an upper bound of the same general type where the limit is raised 50 % above the uniform atmosphere model. The conclusion is that, in the absence of stable layers, the simple formulae for max. g.l.c. can never be exceeded by a factor of more than 20 to 50 %. It is difficult to distinguish such a factor from field trial results owing to the fact that inherent experimental errors have about the same order of magnitude. However, now that techniques for simulating atmospheric boundary layers in wind tunnels are well advanced (Counihan 1969) this result can be tested by realistic controlled experiments. It is important that such checks be made as once these limits are firmly established it means that future field trial work need only concentrate on peculiar effects of topography and meteorology rather than lengthy and costly general surveys. The effects of stable layers have also been assessed, indicating that they can at most increase the above limits by about a further 80 %. In such cases some care may have to be taken over the definition of source height and indeed the effective point source model might not be wholly adequate for their description. The conventional models describing the thermal rise of the plume may also be suspect in these conditions. However, it seems hardly fortuitous that hourly mean values of max. g.l.c. at Tilbury have never exceeded slightly more than twice the value given by the simple approach, a result which agrees with the estimates given here.

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